

NO-A181 914

OBlique PROJECTIONS: FORMULAS ALGORITHMS AND ERROR  
BOUNDS(U) JOHNS HOPKINS UNIV BALTIMORE MD DEPT OF  
ELECTRICAL ENGINEERIN S KAYALAR ET AL 02 JUN 87

1/1

UNCLASSIFIED

JHU/ECE-87/10 N00014-85-K-0255

F/G 12/1

NL

END  
8-87  
DTIC



MICROCOPY RESOLUTION TEST CHART

The Johns Hopkins University

(12)

DTIC FILE COPY

AD-A181 914

OBLIQUE PROJECTIONS:

FORMULAS, ALGORITHMS, AND ERROR BOUNDS<sup>1</sup>

*Selahattin Kayalar and Howard L. Weinert*

Report JHU/ECE - 87/10



DISTRIBUTION STATEMENT A  
Approved for public release  
Distribution Unlimited

**ELECTRICAL  
ENGINEERING  
& COMPUTER  
SCIENCE**

87-6-23-00

(12)

A -  
DTIC  
ELEC.  
JUN 24 1987  
S D

OBLIQUE PROJECTIONS:  
FORMULAS, ALGORITHMS, AND ERROR BOUNDS<sup>1</sup>

*Selahattin Kayalar and Howard L. Weinert*

Report JHU/ECE - 87/10

ABSTRACT

When an orthogonal projection is to be computed using data acquired by distributed sensors, it is often necessary to process each sensor's data locally and then transmit the results to a central facility for final processing. The most efficient way to do this is to compute oblique projections locally. This choice makes the final processing a matter of summing the oblique projections. In this paper we derive new formulas, and iterative algorithms and associated error bounds, for oblique projections in arbitrary Hilbert spaces.

DISTRIBUTION STATEMENT

Approved for public release  
Distribution Unlimited

---

<sup>1</sup>This work was supported by the Office of Naval Research under contract N00014-85-K-0255.

## INTRODUCTION

The solutions to many signal processing problems can be characterized in terms of orthogonal projection in an appropriate Hilbert space. Linear least-squares filtering, smoothing, and prediction are notable examples. Most existing algorithms for orthogonal projections were developed with the assumption that all the data would be available for processing at a central location. In the case of data acquisition by distributed sensors this assumption is not always realistic because constraints on the communication links may prohibit the transmission of the sensors' data to a central facility for processing. In this case the data subsets must be processed locally (at the sensor sites) to produce appropriate summaries which will then be transmitted to a central location and combined to yield the desired orthogonal projection.

The primary issue, then, is to decide what partial results should be computed locally so that the overall problem is solved as efficiently as possible. An obvious choice is to let each partial result be an orthogonal projection onto the subspace determined by the relevant data subset. In this way each processor will be solving a smaller-scale version of the overall problem. However, there is a serious difficulty with this choice.

To illustrate, let  $V$  be a Hilbert space and let  $H$  be the closed subspace of  $V$  spanned by the data. Suppose the data are partitioned into two subsets. Then these subsets individually

span two closed subspaces  $H_1$  and  $H_2$  such that  $H = H_1 + H_2$ . Suppose  $H_1 \cap H_2 = \{0\}$  and let the orthogonal projection operators onto  $H$ ,  $H_1$  and  $H_2$  be denoted by  $P$ ,  $P_1$  and  $P_2$ , respectively. For a vector  $z \in V$ , if we decide to find  $Pz$  by first computing  $P_1 z$  and  $P_2 z$  locally, then we must combine these partial results according to the following formula due to Aronszajn [A2]:

$$Pz = (I - P_2)(I - P_1 P_2)^{-1} P_1 z + (I - P_1)(I - P_2 P_1)^{-1} P_2 z.$$

It is clear that the operations needed to combine  $P_1 z$  and  $P_2 z$  are more complex than those required to compute either of these projections, thus producing a computational bottleneck at the central facility. We note that all two-filter smoothing formulas are special cases of the above formula. Adamyan and Arov [AA], Salehi [S2], and Pavon [P] have applied this formula directly to solve certain stochastic interpolation problems. As an example of how it specializes in the case of decentralized Kalman filtering, see Speyer [S4]. We note that there is no generalization of Aronszajn's formula to the case of more than two subspaces.

In fact, the bottleneck inherent in the above approach can be avoided by computing oblique, rather than orthogonal, projections at the local sites. This choice makes the combination of the partial results as simple as possible, namely one of straight addition. As will be discussed in more generality later, the direct sum decomposition  $H = H_1 \oplus H_2$  uniquely determines two oblique projection operators  $L_1$  and  $L_2$  such that

$$Pz = L_1 z + L_2 z.$$

In comparing this formula with Aronszajn's given previously, it should be noted that the right-hand sides do not correspond term by term since  $L_1 z$  and  $L_2 z$  lie in  $H_1$  and  $H_2$ , respectively.

It is the purpose of this paper to derive new formulas, and iterative algorithms and associated error bounds, for oblique projections in arbitrary Hilbert spaces. Although some results along these lines exist for decompositions involving only two subspaces (two data subsets) [A1], [A2], [G], [TYM], [Y], there are no such results for decompositions involving more than two subspaces. We should note that in certain more specific settings, it is possible to determine oblique projections using the concept of superposition [RW], [WC]. Along these lines, Chang [C] was able to rearrange the computations in Speyer's decentralized Kalman filtering algorithm so that oblique projections were computed and then summed. He was, however, unaware of the geometric interpretations of his algebraic manipulations. Finally, the concept of oblique projection has been applied in various other contexts [HB], [KD], [S1], [S3], [TYM], [Y].

Accession For	
NTIS CRA&I	<input checked="checked" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and/or Special
A-1	



# OBLIQUE PROJECTION FORMULAS

Let  $H$  and  $H_i$  ( $i=1,2,\dots,k$ ) be closed subspaces of a Hilbert space  $V$  such that  $H$  has the *direct sum* decomposition

$$H = H_1 \oplus H_2 \oplus \dots \oplus H_k.$$

Let the orthogonal projection operators onto  $H$  and  $H_i$  be  $P$  and  $P_i$ , respectively. Let us also introduce the notation  $H_{(i)}$  to mean the direct sum of the  $k-1$  subspaces excluding  $H_i$ , that is,

$$H_{(i)} = H_1 \oplus \dots \oplus H_{i-1} \oplus H_{i+1} \oplus \dots \oplus H_k.$$

Since for any  $z \in V$ ,  $Pz$  is an element of  $H$ , it can be decomposed uniquely as

$$Pz = z_1 + z_2 + \dots + z_k$$

where  $z_i \in H_i$  ( $i=1,2,\dots,k$ ). The map that takes  $z$  to  $z_i$  determines a unique linear idempotent operator  $L_i$  called the *oblique projection operator* onto  $H_i$  along  $H_{(i)} \oplus H^\perp$ , where  $H^\perp$  is the orthogonal complement of  $H$ . Thus

$$Pz = L_1 z + L_2 z + \dots + L_k z.$$

The range and null space of  $L_i$  are  $H_i$  and  $H_{(i)} \oplus H^\perp$ , respectively. Note that

$$L_i L_j = 0, \quad i \neq j.$$

The earliest discussions of oblique projection operators appear to be those in Murray[M] and Lorch[L1]. Other useful



treatments can be found in Afriat[A1], Halmos[H1], Kato[K], Lyantse[L2], and Takeuchi *et al* [TYM].

For  $i=1,2,\dots,k$ , let  $M_i = H_i^\perp \cap H$ , and let  $\pi_i$  be the orthogonal projection operator onto  $M_i$ . It is clear that  $\pi_i = P - P_i$ . Let  $\pi_{i \downarrow j}$  be defined as

$$\pi_{i \downarrow j} = \pi_i \pi_{i-1} \cdots \pi_j$$

for  $1 \leq j \leq i \leq k$ , and  $\pi_{i \downarrow j} = I$  whenever  $i < j$ . In addition, let

$$\pi_{i \downarrow} = \begin{cases} \pi_{k \downarrow 2}, & i=1, \\ \pi_{k-1 \downarrow 1}, & i=k, \\ \pi_{i-1 \downarrow 1} \pi_{k \downarrow i+1}, & 2 \leq i \leq k-1. \end{cases}$$

Theorem 1: The oblique projection operator  $L_i$  ( $i=1,2,\dots,k$ ) is given by

$$L_i = \begin{cases} P_i \pi_{i-1 \downarrow j+1} (I - \pi_j \pi_{j \downarrow})^{-1}, & 1 \leq j \leq i-1, \\ P_i \pi_{i-1 \downarrow 1} \pi_{k \downarrow j+1} (I - \pi_j \pi_{j \downarrow})^{-1}, & i \leq j \leq k. \end{cases}$$

**Proof:** Note that this theorem gives a different representation of the oblique projection operator  $L_i$  for each  $j=1,2,\dots,k$ . We will prove only the representation for  $j=k$ . The other representations follow in a similar manner.

Using  $\pi_i = P - P_i$  for  $i=1,2,\dots,k$ , we see that

$$\pi_k \pi_{k \downarrow} = \pi_{k \downarrow 1} = (P - P_k) \pi_{k-1 \downarrow 1} = \pi_{k-1 \downarrow 1} - P_k \pi_{k-1 \downarrow 1}.$$

since  $P\pi_{k-1\downarrow 1} = \pi_{k-1\downarrow 1}$ . Similarly, the first term on the right can be expanded as

$$\pi_{k-1\downarrow 1} = (P - P_{k-1})\pi_{k-2\downarrow 1} = \pi_{k-2\downarrow 1} - P_{k-1}\pi_{k-2\downarrow 1}.$$

Combining these two relations, we have

$$\pi_k \pi_{k\downarrow} = \pi_{k-2\downarrow 1} - P_{k-1}\pi_{k-2\downarrow 1} - P_k \pi_{k-1\downarrow 1}.$$

Expanding  $\pi_{k-2\downarrow 1}$  and continuing in the same manner, we get

$$\pi_k \pi_{k\downarrow} = P - P_1 \pi_{0\downarrow 1} - P_2 \pi_{1\downarrow 1} - \dots - P_k \pi_{k-1\downarrow 1}.$$

We can rearrange this equation as follows:

$$P - \pi_k \pi_{k\downarrow} = P(I - \pi_k \pi_{k\downarrow}) = P_1 \pi_{0\downarrow 1} + P_2 \pi_{1\downarrow 1} + \dots + P_k \pi_{k-1\downarrow 1}.$$

Since by Lemma A5  $(I - \pi_k \pi_{k\downarrow})^{-1}$  exists, the result of the theorem for  $j=k$  follows.  $\square$

There is another class of representations for oblique projection operators that uses the inverses of selfadjoint operators. Let us introduce the following additional notation:

$$\pi_{i\uparrow j} = \pi_i \pi_{i+1} \dots \pi_j,$$

for  $1 \leq i \leq j \leq k$  and  $\pi_{i\uparrow j} = I$  whenever  $i > j$ . Furthermore, let

$$\pi_{i\uparrow} = \begin{cases} \pi_{2\uparrow k}, & i=1, \\ \pi_{1\uparrow k-1}, & i=k, \\ \pi_{i+1\uparrow k} \pi_{1\uparrow i-1}, & 2 \leq i \leq k-1. \end{cases}$$

Theorem 2: The oblique projection operator  $L_i$  ( $i=1,2,\dots,k$ ) is given by

$$L_i = P_i \pi_{i\downarrow} (I - \pi_{i\uparrow} \pi_i \pi_{i\downarrow})^{-1}.$$

Proof: Since  $\pi_i$  is a factor of neither  $\pi_{i\uparrow}$  nor  $\pi_{i\downarrow}$ , we can write, as in the proof of Theorem 1,

$$\pi_{i\uparrow} \pi_i \pi_{i\downarrow} = P - P_i \pi_{i\downarrow} - \Gamma,$$

where  $\Gamma$  denotes the sum of the remaining terms, none of which have  $P_i$  as a left factor. Now

$$P - \pi_{i\uparrow} \pi_i \pi_{i\downarrow} = P(I - \pi_{i\uparrow} \pi_i \pi_{i\downarrow}) = P_i \pi_{i\downarrow} + \Gamma,$$

and

$$P = P_i \pi_{i\downarrow} (I - \pi_{i\uparrow} \pi_i \pi_{i\downarrow})^{-1} + \Gamma (I - \pi_{i\uparrow} \pi_i \pi_{i\downarrow})^{-1}.$$

The necessary inverse exists by Lemma A5. This decomposition clearly indicates that

$$L_i = P_i \pi_{i\downarrow} (I - \pi_{i\uparrow} \pi_i \pi_{i\downarrow})^{-1}. \quad \square$$

Note that  $\pi_{i\uparrow}^* = \pi_{i\downarrow}$ , where  $\pi_{i\uparrow}^*$  is the adjoint of  $\pi_{i\uparrow}$ . Therefore  $\pi_{i\uparrow} \pi_i \pi_{i\downarrow}$  is selfadjoint. One can also obtain the result of Theorem 2 directly from Theorem 1 (with  $j=i$ ) using the easily established fact that  $L_m \pi_n = L_m$ ,  $m \neq n$ .

## SERIES EXPANSIONS AND ITERATIVE METHODS

### I. The Case of Two Subspaces

For the case of two subspaces the oblique projection operators given by Theorem 1 (with  $j=i$ ) are

$$\begin{aligned}L_1 &= P_1 \pi_2 (I - \pi_1 \pi_2)^{-1}, \\L_2 &= P_2 \pi_1 (I - \pi_2 \pi_1)^{-1}.\end{aligned}$$

These expressions can be written in a form more conducive to iterative implementation. The following lemmas are basic in this direction.

Lemma 1: Under the stated assumptions on  $H_1$  and  $H_2$ ,

$$\|P_2 P_1\| = \|P_1 P_2\| = \|\pi_2 \pi_1\| = \|\pi_1 \pi_2\| < 1.$$

Proof: Since  $H_2 \cap H_1 = \{0\}$  and  $M_2 \cap M_1 = \{0\}$ , we have

$$\cos \varphi(H_2, H_1) = \cos \varphi_c(H_2, H_1),$$

and

$$\cos \varphi(M_2, M_1) = \cos \varphi_c(M_2, M_1).$$

These angles are defined in the Appendix. From Lemma A1 we have the equality

$$\cos \varphi_c(H_2, H_1) = \cos \varphi_c(M_2, M_1).$$

In addition, it is well-known (see, for example, Youla [Y]) that

$$\|P_2 P_1\| = \|P_1 P_2\| = \cos \varphi(H_2, H_1),$$

and

$$\|\pi_2 \pi_1\| = \|\pi_1 \pi_2\| = \cos \varphi(M_2, M_1).$$

Since  $H_1 \oplus H_2$  is closed, we have  $\varphi_c(H_2, H_1) > 0$  by Lemma A2. Now the result follows.  $\square$

Lemma 2:  $P_1(\pi_2 \pi_1)^n \pi_2 = (P_1 P_2)^n P_1 \pi_2$  for  $n=1, 2, \dots$ .

Proof: For  $n=1$ , we have

$$P_1 \pi_2 \pi_1 \pi_2 = P_1 (P - P_2)(P - P_1) \pi_2 = P_1 P_2 P_1 \pi_2.$$

Assume that the lemma is true for  $n=m$ . Then for  $n=m+1$ , we have

$$\begin{aligned} P_1(\pi_2 \pi_1)^{m+1} \pi_2 &= P_1(\pi_2 \pi_1)^m \pi_2 \pi_1 \pi_2 \\ &= (P_1 P_2)^m P_1 \pi_2 \pi_1 \pi_2 \\ &= (P_1 P_2)^{m+1} P_1 \pi_2. \end{aligned}$$

Therefore the lemma is true for  $n=m+1$ , and thus it is true for all  $n$ .  $\square$

Theorem 3: For two subspaces the oblique projection operators are given by

$$L_1 = (I - P_1 P_2)^{-1} P_1 Q_2,$$

and

$$L_2 = (I - P_2 P_1)^{-1} P_2 Q_1,$$

with  $Q_i = I - P_i$ .

Proof: We start with the oblique projection formula

$$L_1 = P_1 \pi_2 (I - \pi_1 \pi_2)^{-1}.$$

Since by Lemma 1  $\|\pi_1 \pi_2\| < 1$ , the above inverse can be expanded in an absolutely convergent von Neumann series [K] to give

$$L_1 = \sum_{i=0}^{\infty} P_1 \pi_2 (\pi_1 \pi_2)^i = \sum_{i=0}^{\infty} P_1 (\pi_2 \pi_1)^i \pi_2.$$

Now if we apply Lemma 2 to each term of the series, we get

$$L_1 = \sum_{i=0}^{\infty} (P_1 P_2)^i P_1 \pi_2.$$

From Lemma A5 and Lemma 2,  $(I - P_1 P_2)^{-1}$  exists and

$$(I - P_1 P_2)^{-1} = \sum_{i=0}^{\infty} (P_1 P_2)^i.$$

Therefore

$$L_1 = (I - P_1 P_2)^{-1} P_1 \pi_2.$$

Now the result follows from the fact that  $P_1 \pi_2 = P_1 Q_2$ . The formula for  $L_2$  can be derived similarly.  $\square$

Note that we started with the formulas given by Theorem 1 for  $j=1$ . In fact, all of the other formulas given in Theorem 1 and Theorem 2 will lead to the same expressions given in Theorem 3.

The formulas just derived were previously obtained by Aronszajn [A2] in series form, and by Afriat [A1] in operator form using other methods. The above derivation clearly points out the special structure that exists in the case of two subspaces. As we will see later, this state of affairs has no analog in the case of

three or more subspaces. A similar phenomenon was apparent in our previous work [KW]. We also note here that Greville [G] has derived other oblique projection formulas for the special case of complementary subspaces ( $V \neq H$ ).

The formulas in Theorem 3, in conjunction with the series expansion of  $(I - P_1 P_2)^{-1}$ , lead to an iterative oblique projection algorithm suitable for decentralized problems. We will work with  $L_1$ , but all the results apply to  $L_2$  as well with the obvious changes of subscripts.

From Theorem 3 we have

$$L_1 = \sum_{i=0}^{\infty} (P_1 P_2)^i P_1 Q_2.$$

Let  $L_1^{(n)}$  be the truncated version

$$L_1^{(n)} = \sum_{i=0}^{n-1} (P_1 P_2)^i P_1 Q_2.$$

It is clear that  $\lim_{n \rightarrow \infty} L_1^{(n)} = L_1$ . Therefore the sequence

$$x^{(n)} = L_1^{(n)} z, \quad n=1, 2, \dots$$

converges to  $z_1 = L_1 z$ , the oblique projection of  $z$  onto  $H_1$  along  $H_2 \oplus H^\perp$ . By breaking off the first term in the formula for  $L_1^{(n)}$ , we get

$$L_1^{(n)} z = P_1 Q_2 z + P_1 P_2 L_1^{(n-1)} z.$$

Thus we have the iteration

$$x^{(n)} = P_1 P_2 x^{(n-1)} + P_1 Q_2 z, \quad n=1, 2, \dots$$

with  $x^{(0)} = 0$ . Clearly,  $L_1 z$  is the fixed point of the map

$$F: u \mapsto P_1 P_2 u + P_1 Q_2 z, \quad u \in V.$$

Since  $\|P_1 P_2\| < 1$ ,  $F$  is strictly contractive. Thus, the approximating sequence

$$x^{(n)} = F x^{(n-1)}$$

will converge to  $L_1 z$  for any starting point in  $V$ . Furthermore, any other fixed point algorithm can be used to generate the oblique projection.

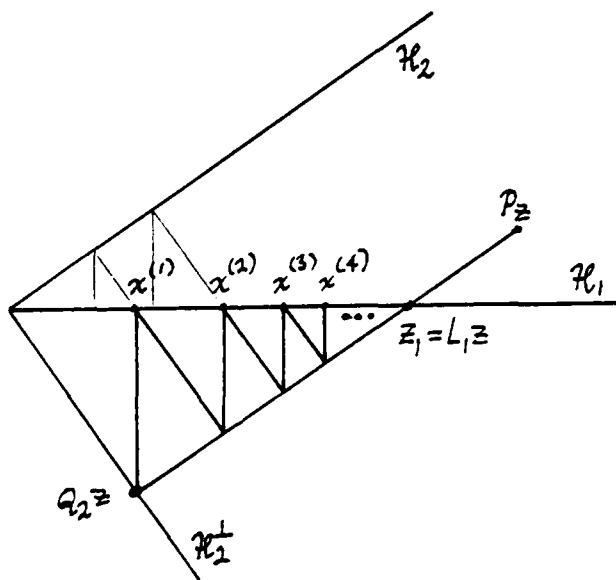


Figure 1

Figure 1 provides a sketch of how the iteration works. The



element  $z \in V$  is first projected orthogonally onto  $H_2^\perp$  to form  $Q_2 z$ , which is then projected orthogonally onto  $H_1$  to obtain  $x^{(1)}$ . Then, the operator  $P_1 P_2$  is applied to  $x^{(1)}$  and the result is added to  $x^{(1)}$  to produce  $x^{(2)}$ . Next, the operator  $P_1 P_2$  is applied to  $x^{(2)}$  and the result is added to  $x^{(1)}$  to produce  $x^{(3)}$ . The iterates get progressively closer to the point  $z_1$ .

It is important to note that our iterative algorithm has the property that each  $x^{(n)}$  is an element of  $H_1$ . This is critical for decentralized problems in which, typically, the processor computing  $z_1$  has access only to the data set spanning  $H_1$ . The iterative oblique projection algorithm of Youla[Y], which is valid only in the complementary subspaces case ( $V=H$ ), does not share this property.

As with any iterative procedure, it is useful to have a good a priori error bound so that the number of iterations needed to achieve a desired accuracy can be reliably estimated. To this end we will need the norm of the oblique projection operator. The result of the next lemma is due to Aronszajn[A2]. It was rediscovered by Del Pasqua[D], Lyantse[L2], and by Youla[Y], who gives an accessible proof for complementary subspaces. The extension to non-complementary subspaces is straightforward.

Lemma 3: The norms of the oblique projection operators are given by

$$\|L_1\| = \|L_2\| = 1/s,$$

where  $s = \sin \varphi(H_2, H_1)$ .

Now, the error operator for the iterative algorithm is given by

$$\begin{aligned} E_1^{(n)} &\triangleq L_1 - L_1^{(n)} = \sum_{i=n}^{\infty} (P_1 P_2)^i P_1 Q_2 = \sum_{i=n}^{\infty} (P_1 P_2 P_1)^i P_1 Q_2 \\ &= (P_1 P_2 P_1)^n \sum_{i=0}^{\infty} (P_1 P_2 P_1)^i P_1 Q_2 \\ &= (P_1 P_2 P_1)^n L_1. \end{aligned}$$

The next theorem gives the sharpest known upper bound on the norm of the error operator.

Theorem 4:  $\|E_1^{(n)}\| = \|(P_1 P_2 P_1)^n L_1\| \leq c^{2n}/s$

where  $c = \cos \varphi(H_2, H_1)$  and  $s = \sin \varphi(H_2, H_1)$ .

Proof: Since  $P_1 P_2 P_1$  is selfadjoint

$$\|(P_1 P_2 P_1)^n\| = \|P_1 P_2 P_1\|^n.$$

Therefore

$$\|(P_1 P_2 P_1)^n L_1\| \leq \|P_1 P_2 P_1\|^n \|L_1\|.$$

The desired result follows from Lemma 3 and the well-known fact that  $\|P_1 P_2 P_1\| = c^2$ .  $\square$

A looser bound, namely  $c^{n-1}/s$ , was given by Aronszajn[A2].

## II. Three or More Subspaces

In the case of three or more subspaces, the oblique projection operators can also be expressed in terms of infinite series. To this end we need the following lemma which is analogous to Lemma 1.

Lemma 4: If the sum of any number of the subspaces  $H_1, H_2, \dots, H_k$  is closed, then

$$\|\pi_i \pi_{i+1}\| = \|\pi_{i+1} \pi_i\| < 1$$

for  $i=1, 2, \dots, k$ .

Proof: Here we will prove the lemma for the case of  $i=k$ . The others will follow similarly.

Let

$$M_{k:j} = M_k \cap M_{k-1} \cap \dots \cap M_j$$

for  $j=1, 2, \dots, k$ . From our earlier work on error bounds for the method of alternating projections [KW], and [HS], we have

$$\|\pi_k \pi_{k+1}\| = \|\pi_{k+1} \pi_k\| = \|\pi_k \dots \pi_2 \pi_1\| \leq \sqrt{1 - s_{k:k}^2 s_{k:k-1}^2 \dots s_{k:2}^2},$$

where  $s_{k:j} = \sin \varphi_c(M_{k:j}, M_{j-1})$  for  $j=2, 3, \dots, k$ . Using Lemma A1 we get

$$s_{k:j} = \sin \varphi_c(H_k \oplus H_{k-1} \oplus \dots \oplus H_j, H_{j-1}).$$

Since by our hypothesis  $H_k \oplus H_{k-1} \oplus \dots \oplus H_j \oplus H_{j-1}$  is closed, we have  $s_{k:j} > 0$  for  $j=2, 3, \dots, k$  by Lemma A2. This proves the lemma for  $i=k$ .  $\square$

We can now present the series expansions for the oblique projection operators.

Theorem 5: Under the hypothesis of Lemma 4, the oblique projection operator  $L_i$  ( $i=1,2,\dots,k$ ) can be written as

$$L_i = \sum_{m=0}^{\infty} P_i (\pi_{i\downarrow} \pi_i)^m \pi_{i\downarrow}.$$

Proof: From Theorem 1 (with  $j=i$ ) we get

$$L_i = P_i \pi_{i\downarrow} (I - \pi_i \pi_{i\downarrow})^{-1}.$$

Now Lemma 4 allows us to expand the inverse operator in a von Neumann series, as follows:

$$L_i = \sum_{m=0}^{\infty} P_i \pi_{i\downarrow} (\pi_i \pi_{i\downarrow})^m = \sum_{m=0}^{\infty} P_i (\pi_{i\downarrow} \pi_i)^m \pi_{i\downarrow}. \quad \square$$

Since, in the case of three or more subspaces, there is no result analogous to Lemma 2, we cannot derive an iterative oblique projection algorithm like the one presented earlier for the case of two subspaces. Instead, we must compute each term in a truncated version of the infinite series. Again, each term in the series for  $L_i$  is an element of  $H_i$ , and can therefore be computed using only the data set which spans  $H_i$ .

In order to decide on how many terms to retain in the

series, we need an upper bound on the error operator  $E_i^{(n)}$ , where

$$\begin{aligned} E_i^{(n)} &\triangleq L_i - L_i^{(n)} = \sum_{m=n}^{\infty} P_i \Pi_{i\downarrow} (\Pi_i \Pi_{i\downarrow})^m \\ &= \sum_{m=0}^{\infty} P_i \Pi_{i\downarrow} (\Pi_i \Pi_{i\downarrow})^m (\Pi_i \Pi_{i\downarrow})^n \\ &= L_i (\Pi_i \Pi_{i\downarrow})^n. \end{aligned}$$

As before, we need to evaluate the norm of the oblique projection operator in order to derive an upper bound for the norm of the error operator.

Lemma 5: The norm of the oblique projection operator  $L_i$  ( $i=1,2,\dots,k$ ) is given by

$$\|L_i\| = 1/s_i$$

where  $s_i = \sin \varphi(H_i, H_{\langle i \rangle})$ .

Proof: Since we can write  $H = H_i \oplus H_{\langle i \rangle}$  for  $i=1,2,\dots,k$ , Lemma 3, which dealt with two subspaces, applies with the obvious identifications.  $\square$

A bound on the norm of the error operator will be given only for  $i=k$  to avoid introducing new notation. One can always rename the subspaces so that the following theorem applies.

Theorem 6:  $\|E_k^{(n)}\| = \|L_k(\pi_k \pi_{k\downarrow})^n\| \leq \kappa_k^n / s_k$ ,

where

$$s_k = \sin \varphi(H_k, H_{\langle k \rangle}),$$

$$\kappa_k^2 = 1 - s_{k:k}^2 s_{k:k-1}^2 \cdots s_{k:2}^2,$$

and for  $j=2,3,\dots,k$

$$s_{k:j} = \sin \varphi(H_k \oplus H_{k-1} \oplus \cdots \oplus H_j, H_{j-1}).$$

Proof: Since  $\|L_k(\pi_k \pi_{k\downarrow})^n\| \leq \|L_k\| \|\pi_k \pi_{k\downarrow}\|^n$ , we need to derive an upper bound for  $\|\pi_k \pi_{k\downarrow}\|$ . We know from the proof of Lemma 4 that

$$\|\pi_k \pi_{k\downarrow}\|^2 \leq 1 - s_{k:k}^2 s_{k:k-1}^2 \cdots s_{k:2}^2$$

where

$$s_{k:j} = \sin \varphi_c(H_k \oplus H_{k-1} \oplus \cdots \oplus H_j, H_{j-1}).$$

Since

$$(H_k \oplus H_{k-1} \oplus \cdots \oplus H_j) \cap H_{j-1} = \{0\}$$

we have

$$\varphi_c(H_k \oplus H_{k-1} \oplus \cdots \oplus H_j, H_{j-1}) = \varphi(H_k \oplus H_{k-1} \oplus \cdots \oplus H_j, H_{j-1}).$$

Now the result follows by an application of Lemma 5.  $\square$

An even sharper upper bound can be derived by applying the fundamental inequalities in Kayalar and Weinert[KW] to  $\|(\pi_k \pi_{k\downarrow})^n\|$ .

## APPENDIX

In this appendix we will first state the definitions of minimum and complementary angles, and then give some basic results concerning angles and products of projections.

Let  $H_1$  and  $H_2$  be closed subspaces of a Hilbert space  $V$ .

Definition A1: The *minimum angle*  $\varphi(H_2, H_1)$  between the subspaces  $H_2$  and  $H_1$  is the angle between 0 and  $\pi/2$  whose cosine is given by

$$\cos \varphi(H_2, H_1) = \sup\{ |\langle y, x \rangle| : y \in H_2, x \in H_1, \|y\| \leq 1, \|x\| \leq 1 \},$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $V$ .

If we remove the intersection from the two subspaces, we get a slightly different definition. Let  $H_{2:1} \triangleq H_2 \cap H_1^\perp$ .

Definition A2: The *complementary angle*  $\varphi_c(H_2, H_1)$  between  $H_2$  and  $H_1$  is

$$\varphi_c(H_2, H_1) = \varphi(H_2 \cap H_{2:1}^\perp, H_1 \cap H_{2:1}^\perp).$$

Note that if  $H_2 \cap H_1 = \{0\}$ , then  $\varphi_c(H_2, H_1) = \varphi(H_2, H_1)$ .

Lemma A1: Let  $H$  and  $\{H_i\}_{i=1}^k$  be closed subspaces of a Hilbert space  $V$ , with

$$H = H_1 \oplus H_2 \oplus \dots \oplus H_k$$

and

$$M_{i:j} = H_i^\perp \cap H_{i-1}^\perp \cap \dots \cap H_j^\perp \cap H.$$

Then

$$\varphi_c(M_{i:j}, M_{j-1}) = \varphi_c(H_i \oplus H_{i-1} \oplus \dots \oplus H_j, H_{j-1}),$$

for  $2 \leq j \leq k$ .

Proof: Since  $M_i = H_i^\perp \cap H$ ,

$$\begin{aligned} \varphi_c(M_{i:j}, M_{j-1}) &= \varphi_c(H_i^\perp \cap H_{i-1}^\perp \cap \dots \cap H_j^\perp \cap H, H_{j-1}^\perp \cap H) \\ &= \varphi_c((H_i \oplus H_{i-1} \oplus \dots \oplus H_j)^\perp \cap H, H_{j-1}^\perp \cap H). \\ &= \varphi_c((H_i \oplus H_{i-1} \oplus \dots \oplus H_j)^\perp \cap H \oplus H^\perp, H_{j-1}^\perp \cap H \oplus H^\perp) \\ &= \varphi_c((H_i \oplus H_{i-1} \oplus \dots \oplus H_j)^\perp, H_{j-1}^\perp) \\ &= \varphi_c(H_i \oplus H_{i-1} \oplus \dots \oplus H_j, H_{j-1}). \quad \square \end{aligned}$$

The result stated in the next lemma is due to Lorch[L1], but has been rediscovered many times.

Lemma A2: Let  $H_1$  and  $H_2$  be closed subspaces of a Hilbert space  $V$ . Then

$$\varphi_c(H_2, H_1) > 0$$

if and only if  $H_2 + H_1$  is closed.

Now let  $H$  and  $H_i$  ( $i=1, 2, \dots, k$ ) be closed subspaces of a



Hilbert space  $V$  such that

$$H = H_1 + H_2 + \dots + H_k,$$

where this decomposition is not necessarily a direct sum decomposition. Let  $H_{k:1} = H_k \cap \dots \cap H_2 \cap H_1$ , and let the orthogonal projection operators onto  $H$ ,  $H_i$  and  $H_{k:1}$  be  $P$ ,  $P_i$  and  $P_{k:1}$ , respectively. In the lemmas below, let

$$\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, k\}.$$

The following result is due to Halperin [H2].

Lemma A3:  $\lim_{n \rightarrow \infty} (P_{i_m} \dots P_{i_2} P_{i_1})^n = P_{k:1}.$

The next result was proved by Afriat[A1] for the case of two subspaces.

Lemma A4:  $N\left[I - P_{i_m} \dots P_{i_2} P_{i_1}\right] = H_{k:1}$ , where  $N(\cdot)$  denotes the null space of the indicated operator.

Proof: It is easy to verify that

$$H_{k:1} \subset N\left[I - P_{i_m} \dots P_{i_2} P_{i_1}\right].$$

Now let  $x \in N\left[I - P_{i_m} \dots P_{i_2} P_{i_1}\right]$ . Then we have  $x = P_{i_m} \dots P_{i_2} P_{i_1} x$ , which implies

$$x = (P_{i_m} \dots P_{i_2} P_{i_1})^n x,$$

for all  $n$ . Taking the limit as  $n \rightarrow \infty$  and using lemma A3, we see that  $x = P_{k:1} x$ . This shows that  $x \in H_{k:1}$ , and thus

$$\mathcal{N} \left[ I - P_{i_m} \cdots P_{i_2} P_{i_1} \right] \subset H_{k:1}.$$

Now the result follows.  $\square$

Now recall that  $M_i = H_i^\perp \cap H$ , and  $\pi_i$  is the orthogonal projection operator onto  $M_i$ .

Lemma A5: The operator  $(I - \pi_{i_m} \cdots \pi_{i_2} \pi_{i_1})$  is invertible.

Proof: From lemma A4 we have

$$\mathcal{N} \left[ I - \pi_{i_m} \cdots \pi_{i_2} \pi_{i_1} \right] = M_{k:1}.$$

Since

$$H_k^\perp \cap \cdots \cap H_2^\perp \cap H_1^\perp = (H_k + \cdots + H_2 + H_1)^\perp,$$

we have

$$\begin{aligned} M_{k:1} &= H_k^\perp \cap \cdots \cap H_2^\perp \cap H_1^\perp \cap H \\ &= (H_k + \cdots + H_2 + H_1)^\perp \cap H \\ &= H^\perp \cap H \\ &= \{0\}. \end{aligned}$$

This proves the lemma.  $\square$

# REFERENCES

- [A1] Afriat, S.N., "Orthogonal and oblique projectors and the characteristics of pairs of vector spaces," *Proc. Cambridge Philos. Soc.*, 53 (1957), 800-816.
- [A2] Aronszajn, N., "Theory of reproducing kernels," *Trans. Amer. Math. Soc.*, 68 (1950), 337-404.
- [AA] Adamyan, V.M. and D.Z. Arov, "A general solution of a problem in linear prediction of stationary processes," *Theory Prob. Appl.*, 13 (1968), 394-407.
- [C] Chang, T.-S., "Comments on 'Computation and transmission requirements for a decentralized linear-quadratic-Gaussian control problem'," *IEEE Trans. Automatic Control*, AC-25 (1980), 609-610.
- [D] Del Pasqua, D., "On a concept of disjoint linear manifolds in a Banach space," *Rendiconti di Matematica*, 13 (1955), 406-422.
- [G] Greville, T.N.E., "Solutions of the matrix equation  $XAX=X$ , and relations between oblique and orthogonal projectors," *SIAM J. Appl. Math.*, 26 (1974), 828-832.

- [H1] Halmos, P.R., *Finite Dimensional Vector Spaces*, Springer, New York, 1974.
- [H2] Halperin, I., "The product of projection operators," *Acta Sci. Math.*, 23 (1962), 96-99.
- [HB] Hyland, D.C. and D.S. Bernstein, "The optimal projection equations for model reduction and the relationships among the methods of Wilson, Skelton and Moore," *IEEE Trans. Automatic Control*, AC-30 (1985), 1201-1211.
- [HS] Hamaker, C. and D.C. Solmon, "The angles between the null spaces of X-rays," *J. Math. Anal. Appl.*, 62 (1978), 1-23.
- [K] Kato, T., *A Short Introduction to Perturbation Theory for Linear Operators*, Springer, New York, 1982.
- [KD] Klein, J.D. and B.W. Dickinson, "A normalized ladder form of the residual energy ratio algorithm for PARCOR estimation via projections," *IEEE Trans. Automatic Control*, AC-28 (1983), 943-952.
- [KW] Kayalar, S. and H.L. Weinert, "Error bounds for the method of alternating projections," Report JHU/ECE-87/02, The Johns Hopkins University, 1987, to appear in *Mathematics of Control, Signals, and Systems*.

- [L1] Lorch, E.R., "On a calculus of operators in reflexive vector spaces," *Trans. Amer. Math. Soc.*, 45 (1939), 217-234.
  
- [L2] Lyantse, V.E., "Some properties of idempotent operators," *Teoreticheskaya i Prikladnaya Matematika*, 1 (1958), 16-22.
  
- [M] Murray, F.J., "On complementary manifolds and projections in  $L_p$  and  $l_p$ ," *Trans. Amer. Math. Soc.*, 41 (1937), 138-152.
  
- [P] Pavon, M., "New results on the interpolation problem for continuous-time stationary-increments processes," *SIAM J. Control and Optimization*, 22 (1984), 133-142.
  
- [RW] Riddle, L.R. and H.L. Weinert, "Fast algorithms for the reconstruction of images from hyperbolic systems," *Proc. 25th Conference on Decision and Control* (1986), 173-178.
  
- [S1] Saad, Y., "The Lanczos biorthogonalization algorithm and other oblique projection methods for solving large unsymmetric systems," *SIAM J. Numer. Anal.*, 19 (1982), 485-506.
  
- [S2] Salehi, H., "On the alternating projections theorem and bivariate stationary stochastic processes," *Trans. Amer. Math. Soc.*, 128 (1967), 121-134.

- [S3] Samson, C., "A unified treatment of fast algorithms for identification," *Int. J. Control*, 35 (1982), 909-934.
- [S4] Speyer, J.L., "Computation and transmission requirements for a decentralized linear-quadratic-Gaussian control problem," *IEEE Trans. Automatic Control*, AC-24 (1979), 266-269.
- [TYM] Takeuchi, K., H. Yanai, and B.N. Mukherjee, *The Foundations of Multivariate Analysis*, Wiley, New York, 1982.
- [WC] Weinert, H.L. and E.S. Chornoboy, "Smoothing with blackouts," pp. 273-278 of *Modelling and Application of Stochastic Processes*, U.B. Desai, ed., Kluwer, Boston, 1986.
- [Y] Youla, D.C., "Generalized image restoration by the method of alternating projections," *IEEE Trans. Circuits and Systems*, CAS-25 (1978), 694-702.

## REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION <i>Unclassified</i>			1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT  <i>Unlimited</i>	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE				
4. PERFORMING ORGANIZATION REPORT NUMBER(S) <i>JHU/ECE - 87/10</i>			5. MONITORING ORGANIZATION REPORT NUMBER(S)	
6a. NAME OF PERFORMING ORGANIZATION <i>Johns Hopkins University</i>		6b. OFFICE SYMBOL (If applicable)		7a. NAME OF MONITORING ORGANIZATION <i>Office of Naval Research</i>
6c. ADDRESS (City, State and ZIP Code) <i>3400 N. Charles Street Baltimore, Maryland 21218</i>			7b. ADDRESS (City, State and ZIP Code) <i>800 N. Quincy Street Arlington, Virginia 22217-5000</i>	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable)		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER <i>N00014-85-K-0255</i>
8c. ADDRESS (City, State and ZIP Code)			10. SOURCE OF FUNDING NOS.	
			PROGRAM ELEMENT NO.	TASK NO.  <i>NR661-019</i>
11. TITLE (Include Security Classification) <i>Oblique Projections: Formulas, Algorithms, and Error Bounds (Unclassified)</i>				
12. PERSONAL AUTHOR(S) <i>S. Kayalar and H. L. Weinert</i>				
13a. TYPE OF REPORT <i>Interim</i>		13b. TIME COVERED FROM <i>5/1/85</i> TO <i>6/2/87</i>		14. DATE OF REPORT (Yr., Mo., Day) <i>June 2, 1987</i>
15. PAGE COUNT <i>27</i>				
16. SUPPLEMENTARY NOTATION				
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB. GR.	<i>Oblique projections, iterative algorithms, error bounds, decentralized processing</i>	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)				
<p>When an orthogonal projection is to be computed using data acquired by distributed sensors, it is often necessary to process each sensor's data locally and then transmit the results to a central facility for final processing. The most efficient way to do this is to compute oblique projections locally. This choice makes the final processing a matter of summing the oblique projections. In this paper we derive new formulas, and iterative algorithms and associated error bounds, for oblique projections in arbitrary Hilbert spaces.</p>				
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> OTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION <i>Unclassified</i>	
22a. NAME OF RESPONSIBLE INDIVIDUAL <i>Dr. Neil Gerr</i>			22b. TELEPHONE NUMBER (Include Area Code) <i>(202) 696-4321</i>	22c. OFFICE SYMBOL

END

8-87

DTIC